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Stability of a regular polygon of finite vortices

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It is well known that a system of N point vortices arranged in a circular row, so that the vortices are at the vertices of a regular polygon, is stable if $N < 7$, neutrally stable if $N = 7$ and unstable if $N > 7$ (Havelock 1931). The effect on this result of taking account of the finite size of the vortices is considered analytically. The vortices are considered to be uniform with small but finite core. Approximate equations for the shape and motion of a vortex subjected to an external velocity field are given and used to evaluate the shape and angular velocity of rotation of the system and to study its stability to plane infinitesimal disturbances. It is found that the system is stable if $N < 7$ and unstable if $N \geqslant 7$. These asymptotic results for small core area are in general consistent with Dritschel (1985) where the motion and stability of up to $N = 8$ finite vortices is evaluated numerically; the steady configuration and the stability results for these values of N are in agreement except in a region of parameter space where a high degree of accuracy is required in the numerical calculation to resolve the growth rate of small disturbances. The case of a linear array of finite vortices is obtained as a special limiting case of the system. The growth rate of plane infinitesimal disturbances for this case is given.

1. Introduction

In fluid flows where most of the vorticity is confined to small two-dimensional patches or layers, the induced velocity field away from these regions can be approximated by regarding the vorticity regions to be represented by single point vortices or vortex sheets embedded in inviscid and irrotational fluid. The stability of these idealized systems to infinitesimal disturbances may be studied with some physical justification since the possible existence of the represented flow requires that the canonical flow is stable to small perturbations. Such studies date back to Kelvin and Rayleigh.

However, it is possible that such singular distribution of vorticity can give rise to spurious instabilities and it is desirable to consider an improved model of the flow by taking account of the finite size of the patches and layers. In the case of a vortex sheet, it is well known that taking account of the thickness of the vortex layer which it represents suppresses the growth of short-wave disturbances to the sheet. This appears to have sometimes led to the misconception that for any vortex system, greater stability will be predicted if the finite size of the region of vorticity is taken into account than otherwise. The available evidence, however, does not bear this out.

A single row of point vortices is unstable while a staggered row is stable only for

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a particular aspect ratio (Lamb 1975, pp. 225–229). Saffman & Szeto (1981) have shown that a single row of finite-cored uniform vortices is also unstable (in fact, as we shall see in §4, the growth rate of plane disturbances is greater). Further, Meiron, Saffman & Schatzman (1984) have shown that in the case of the staggered array of finite-cored vortices, while finite size stabilizes certain modal instabilities, the array is again unstable except for a particular value of the aspect ratio which depends on the core size. It is believed that similar results will hold if the cores have a non-uniform distribution of vorticity. In the case presented here, finite size has a destabilizing effect.

The study of the stability of linear arrays of vortices is relevant to flows associated with shear layers and wakes behind bluff bodies. The corresponding study of a system of vortices arranged along the circumference of a circle is relevant to stability of the flow associated with a curved shear layer. The stability of a system of N point vortices positioned at the vertices of a regular polygon was first studied by Kelvin and J. J. Thompson (for $N \leqslant 7$). It was, however, Havelock (1931) who proved the result that the system is stable if $N < 7$, neutrally stable if $N = 7$ and unstable if $N > 7$.

In this paper, we consider the effect on this result of taking account of the finite size of the vortex core. Thus we consider the stability of a system of N identical uniform vortices of finite core positioned at the vertices of a N-sided regular polygon. We consider the limiting case in which the area A of the core is small compared to πa^2, where a is the radius of the circle on which the polygon lies and represents a typical length associated with the motion of the vortices.

The system can become unstable through disturbances which either displace each vortex from its quasi-steady state or deform its shape. Here, we shall be solely concerned with the former type of disturbances, this being justified provided that the size of the vortex core is small enough. The change in the displacement of the vortex can be followed by considering the motion of the vortex centroid.

Approximate equations for the motion of the centroid of a uniform vortex of small core when placed in an external velocity field were derived by Kida (1982). In §2, these are rederived using a complex potential formulation; the equations are here obtained to a higher order in core area.

These equations are used in §3 to obtain, analytically, the steady shape of the vortices and angular speed of rotation of the system. In §4, the stability of the system to plane infinitesimal disturbances is considered. It is found that the system is stable for $N < 7$ and unstable for $N \geqslant 7$, the growth rate of disturbances increasing with core size. The limiting case of a linear array of finite uniform vortices is obtained as a special case and the growth rate of disturbances for this case is obtained.

A numerical investigation of this problem for $N \leqslant 8$ vortices was carried out by Dritschel (1985). The shape of the vortices and the period of rotation of the system in steady motion, for these values of N, as obtained by the present approximate method agree well with the corresponding numerical results for vortices of small core size. The angular speed of rotation of the system for two and six vortices is also shown to agree well with the corresponding results obtained using contour dynamics as respectively provided by Melander, Zabusky & Styczek (1986) and a referee. Dritschel considers the stability of the system to displacement-type disturbances as well as those which deform the shape. He finds that displacement-type modes are stable if $N < 7$ and unstable if $N = 7$ or 8, which is consistent with the present results. However, whereas in the present study, the case $N = 7$ is unstable to only one normal mode of disturbance, Dritschel finds that it is unstable to two displacement-type modes. Since in the case $N = 7$ the system of point vortices is neutrally stable, the leading term in the dispersion relation for the growth rate of small disturbances can be quite small for sufficiently small core size and a high degree of computational accuracy is required for resolution of the stability criterion. In order to achieve such an accuracy for small-area vortices, the steady shape of the vortices needs to be evaluated to a high degree of accuracy also. Dritschel does not state the accuracy to which the shapes were evaluated in the specific case of $N = 7$. For $N = 8$ small-area vortices, he needed to evaluate the steady shapes to an accuracy of $O(10^{-7})$ to achieve a four-figure accuracy in growth rate. The present asymptotic results for small-area vortices show that in the particular case of $N = 7$, the demand on accuracy in a numerical calculation would be considerably higher, such that a reliable resolution of the stability criterion cannot be given for vortices of sufficiently small core size. For large-area vortices, the requirement in accuracy in evaluating the steady shape of the vortices is apparently not so high. Comparison between the present asymptotic results and Dritschel's numerical results would then suggest the existence of a threshold core size for the mode in question, such that the mode is unstable for core size larger than the threshold value but is stable otherwise. This is, however, of academic interest only since, according to both results, the system is unstable to the other mode of any non-zero value of core size. For large enough vortices, Dritschel finds that for all values of $N (N > 1)$ considered, the system is unstable to disturbances which distort the shapes.

The stability calculations for small and large vortices suggest how a circular vortex sheet, through the action of Helmholtz instability and viscous decay, could degenerate into a rotating system of $N < 7$ finite vortices on the vertices of a regular polygon. As the vortex becomes large enough through viscous diffusion, the given system becomes unstable and evolves into a stable system of fewer vortices, the process being repeated until a single finite core results. These suggestions are fairly consistent with observations of Weske & Rankin (1963) and have potential implications for the evolution of tornadoes.

2. General theory

In this section the equations governing the two-dimensional motion of a uniform vortex of finite core when subjected to an external irrotational velocity field (U_E, V_E) are derived. (U_E, V_E) is the velocity contribution due to sources other than the vortex, such as other vortices, image vorticity and so on which produce an irrotational velocity field in the vicinity of the vortex. If ω is the vorticity in the core, the circulation Γ round the vortex is given by

$$\Gamma = \iint_{R_A} \omega \, \mathrm{d}x \, \mathrm{d}y = \omega A, \tag{2.1}$$

where R_A is the region occupied by the vortex and A is its area. Since Γ is conserved and ω is constant, the area A is an invariant of motion.

We define a complex variable $z = x + \mathrm{i}y$. The flow in the vicinity of, but exterior to, R_A is irrotational and the flow field at an external point z can be described by a complex potential $W(z) = \varphi + \mathrm{i}\psi$ where

$$W(z) = W_1(z) + W_E(z). \tag{2.2}$$

Here, W_E denotes the contribution due to the external velocity field so that

$U_E - iV_E = dW_E/dz$, while W_I denotes the contribution to the potential due to the vortex and is given by

$$W_I(z) = \frac{\Gamma}{2\pi i A} \iint_{R_A} \log(z-z')\, dx'\, dy'$$

$$= -\frac{\Gamma}{4\pi A} \oint_{R_c} (z-z') \log(z-z')\, d\bar{z}' + C \tag{2.3}$$

after applying the complex Green's theorem for R_c traversed anticlockwise; R_c is the boundary of R_A. C is a constant, a bar denotes complex conjugate and in the integrand, with z fixed, that branch of logarithm is chosen which remains single-valued as R_c is traversed. We define the position of the vortex by its centroid, z_v,

$$z_v = \frac{1}{\Gamma} \iint_{R_A} \omega z\, dx\, dy = \frac{i}{4A} \oint_{R_c} z^2\, d\bar{z} \tag{2.4}$$

after using (2.1) and applying the complex Green's theorem. Then the velocity of the vortex centroid is given by

$$\frac{dz_v}{dt} = \frac{i}{4A}\frac{d}{dt}\int_0^{2\pi} z^2 \frac{\partial \bar{z}}{\partial \theta} d\theta = \frac{i}{2A}\int_0^{2\pi} z\left(\frac{\partial z}{\partial t}\frac{\partial \bar{z}}{\partial \theta} - \frac{\partial \bar{z}}{\partial t}\frac{\partial z}{\partial \theta}\right) d\theta \tag{2.5}$$

after an integration by parts. Here (and subsequently) a bar denotes complex conjugate while θ is a Lagrangian parameter which characterizes a boundary point and the integrand is evaluated on the boundary. Now, Pullin (1981) has shown that for a boundary point of a uniform vortex,

$$\frac{\partial \bar{z}}{\partial t} = -\frac{\Gamma}{4\pi A} \oint_{R_c} \log(z-z')\, d\bar{z}' + \frac{dW_E}{dz}, \tag{2.6}$$

where the choice of the branch of logarithm and the way R_c is prescribed is as in (2.3). On substituting this into (2.5), it can be shown that

$$\frac{dz_v}{dt} = \frac{i}{2A} \oint_{R_c} (W_E - \bar{W}_E)\, dz - \frac{\Gamma}{4\pi A^2} \oint_{R_c} z\, \mathrm{Im}\left(\oint_{R_c} \log(z-z')\, d\bar{z}'\, dz\right), \tag{2.7}$$

where Im refers to the imaginary part. It is shown in Appendix A that the second term on the right-hand side of (2.7) is zero. Further since W_E is analytic, we can use the Green's theorem and the Cauchy–Riemann relations in (2.7) to show that

$$\frac{d\bar{z}_v}{dt} = \frac{i}{2A} \oint_{R_c} W_E\, d\bar{z}, \tag{2.8}$$

where R_c is traversed in an anticlockwise sense. Note that in the absence of any external flow (so that W_E is constant), (2.8) expresses the invariance of the centroid. Equation (2.8) does not appear to be given anywhere else and is believed to be a new result.

Since $W_E(z,t)$ is analytic in the vicinity of the vortex (assuming any singularity in W_E is sufficiently distant from z_v), its value on the boundary can be expressed as a Taylor series about z_v. On substituting this series into (2.8) and using the definitions of A and z_v, we obtain

$$\frac{d\bar{z}_v}{dt} = \left[\frac{dW_E}{dz}\right]_{z_v} + \sum_{n=3}^{\infty} \frac{1}{n!}\left[\frac{d^n W_E}{dz^n}\right]_{z_v} I_n, \tag{2.9}$$

where
$$I_n = \frac{\mathrm{i}}{2A}\oint_{R_c} (z'-z_v)^n \,\mathrm{d}\bar{z}'. \tag{2.10}$$

The series will converge provided $\max(|z'-z_v|) <$ radius of convergence of the Taylor series for W_{E} for z' on the vortex boundary.

Similarly, by expanding the integrand in (2.3) as a Taylor series about $z = z_v$ and using the definitions of A and z_v, we have

$$W_{\mathrm{I}}(z,t) = \frac{\Gamma}{2\pi\mathrm{i}}\left\{\log(z-z_v) - \sum_{n=2}^{\infty} \frac{1}{n(n+1)(z-z_v)^n} I_{n+1}\right\} + C \tag{2.11}$$

on swapping the order of the integration and the summation; convergence of the series is assured if $|z-z_v| \geqslant \max(|z'-z_v|)$ where z' is a point on the boundary.

In both (2.9) and (2.11), for a vortex of small area A, the first term on the right-hand side corresponds to the point-vortex result while the sum represents an $O(A^2)$ correction due to the finite size of the vortex.

Thus for a given instantaneous value of $W_{\mathrm{E}}(z,t)$, the motion of the vortex centroid can be followed and its induced velocity potential determined if the instantaneous position of the vortex boundary is known. Suppose that this is given by

$$z_{v\mathrm{b}}(\theta,t) = z_v(t) + r_{v\mathrm{b}}(\theta,t)\,\mathrm{e}^{\mathrm{i}\theta}. \tag{2.12}$$

The condition that the boundary of the vortex is a material surface implies that

$$\frac{\partial r_{v\mathrm{b}}}{\partial t} + u_\theta(r_{v\mathrm{b}},\theta,t)\frac{\partial r_{v\mathrm{b}}}{\partial \theta} - u_r(r_{v\mathrm{b}},\theta,t) = 0, \tag{2.13}$$

where $u_r(r,\theta,t)$ and $u_\theta(r,\theta,t)$ are the radial and azimuthal components of the velocity relative to the vortex centroid. At $z_{v\mathrm{b}}$, u_r and u_θ are given by

$$(u_r(r_{v\mathrm{b}},\theta,t) - \mathrm{i}u_\theta(r_{v\mathrm{b}},\theta,t))\,\mathrm{e}^{-\mathrm{i}\theta} = \left(\frac{\mathrm{d}W_{\mathrm{I}}}{\mathrm{d}z}\right)_{z_{v\mathrm{b}}} + \left(\frac{\mathrm{d}W_{\mathrm{E}}}{\mathrm{d}z}\right)_{z_{v\mathrm{b}}} - \frac{\mathrm{d}\bar{z}_v}{\mathrm{d}t}. \tag{2.14}$$

For $W_{\mathrm{E}}(z,t)$ analytic in the vicinity of the vortex, the second and third terms on the right-hand side are respectively obtained from the Taylor expansion of W_{E} about z_v and by the expansion (2.9). $W_{\mathrm{I}}(z_{v\mathrm{b}},t)$ is given by (2.3), the series (2.11) being, in general, not convergent on the vortex boundary. Thus, on using integration by parts in (2.3) (see Pullin 1981),

$$(u_r - \mathrm{i}u_\theta)\,\mathrm{e}^{-\mathrm{i}\theta} = \frac{\Gamma}{4\pi A}\oint \frac{\bar{z}-\bar{z}'}{z-z'}\mathrm{d}z' + \sum_{n=1}^{\infty} c_n r_{v\mathrm{b}}^n \mathrm{e}^{\mathrm{i}n\theta} - \sum_{n=2}^{\infty} c_n \frac{I_{n+1}}{n+1}, \tag{2.15}$$

where
$$c_n = \frac{1}{n!}\left(\frac{\mathrm{d}^{n+1}W_{\mathrm{E}}}{\mathrm{d}z^{n+1}}\right)_{z=z_v}. \tag{2.16}$$

We consider the case of a vortex of small area A such that

$$\epsilon = A/\Lambda^2 \ll 1, \tag{2.17}$$

Λ being a typical length associated with the external flow. We assume that in the absence of the external flow the vortex is circular and expand

$$r_{v\mathrm{b}} = \Lambda\epsilon^{\frac{1}{2}}(1 + \epsilon r_3 + \epsilon^{\frac{3}{2}} r_4 + \ldots). \tag{2.18}$$

The absence of the $O(\epsilon)$ term in (2.18) reflects the fact that the velocity induced by a uniform vortex of elliptical shape differs from that of a circular vortex by $O(\epsilon^2)$ (see Meiron *et al.* (1984)). The definition of the area A and the centroid z_v respectively imply that

$$A = \frac{1}{2}\int_0^{2\pi} r_{vb}^2\,\mathrm{d}\theta = \pi\Lambda^2\epsilon \tag{2.19}$$

and

$$\int_0^{2\pi} r_{vb}^3\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = 0. \tag{2.20}$$

A change in the vortex shape occurs on a timescale A/Γ which is small compared with the timescale Λ^2/Γ associated with the external field. It is therefore convenient to introduce a scaled time variable $t = \epsilon T$ and write

$$\frac{\partial}{\partial t} = \frac{1}{\epsilon}\frac{\partial}{\partial T}+\frac{\partial}{\partial t}$$

in (2.13). Then on substituting for r_{vb} from (2.18) and $\boldsymbol{u}(z_{vb}, t; T)$ from (2.15) into (2.13) and equating coefficients of powers of ϵ to zero, we obtain a set of equations for r_3, r_4, r_5 etc.; use is made of the fact that in (2.14), $\mathrm{d}\bar{z}/\mathrm{d}t = (\mathrm{d}W_\mathrm{E}/\mathrm{d}z)_{z_v}+O(\epsilon^2\Gamma/\Lambda)$. We restrict ourselves here to solving for r_3 and r_4. Assuming that the core boundary remains smooth, we expand

$$r_3 = \sum_{\substack{n=-\infty\\ |n|\neq 0,1}}^{\infty} a_n\,\mathrm{e}^{\mathrm{i}n\theta}, \quad r_4 = \sum_{\substack{n=-\infty\\ |n|\neq 0,1}}^{\infty} b_n\,\mathrm{e}^{\mathrm{i}n\theta},$$

where $a_n(t;T)$ and $b_n(t;T)$ are complex and $|n| = 0, 1$ have been excluded in view of (2.19) and (2.20). Substituting for r_3 and r_4 in (2.13)–(2.18), we obtain

$$\left.\begin{aligned} \frac{\partial a_n}{\partial T}+\frac{\mathrm{i}\Gamma}{2\pi\Lambda^2}(n-1)\,a_n &= \delta_{n2}\left(\frac{\mathrm{d}^2W_\mathrm{E}}{\mathrm{d}z^2}\right)_{z_v}, \\ \frac{\partial b_n}{\partial T}+\frac{\mathrm{i}\Gamma}{2\pi\Lambda^2}(n-1)\,b_n &= \delta_{n3}\frac{\Lambda}{2}\left(\frac{\mathrm{d}^3W_\mathrm{E}}{\mathrm{d}z^3}\right)_{z_v}, \end{aligned}\right\} \tag{2.21}$$

where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. On solving (2.21) for a_n and b_n and hence determining r_3 and r_4, we obtain, after reintroducing A and t in (2.18),

$$\begin{aligned} r_{vb} = \left(\frac{A}{\pi}\right)^{\frac{1}{2}}\Bigg(1+\frac{2A}{\Gamma}\mathrm{Im}\left[\left(\frac{\mathrm{d}^2W_\mathrm{E}}{\mathrm{d}z^2}\right)_{z_v}\mathrm{e}^{2\mathrm{i}\theta}+\sum_{\substack{n\\ |n|\neq 0,1}} c_n\,\mathrm{e}^{\mathrm{i}(n\theta-(n-1)\Gamma t/2A)}\right] \\ +\frac{A^{\frac{3}{2}}}{2\pi^{\frac{1}{2}}\Gamma}\mathrm{Im}\left[\left(\frac{\mathrm{d}^3W_\mathrm{E}}{\mathrm{d}z^3}\right)_{z_v}\mathrm{e}^{3\mathrm{i}\theta}+\sum_{\substack{n\\ |n|\neq 0,1}} d_n\,\mathrm{e}^{\mathrm{i}(n\theta-(n-1)\Gamma t/2A)}\right]+O(A^2/\Lambda^4)\Bigg), \end{aligned} \tag{2.22}$$

where c_n and d_n are complex constants to be determined from initial conditions and the derivatives are evaluated at the centroid z_v. The summations represent homogeneous solutions of (2.21) and correspond to free oscillations of a circular vortex, travelling around the vortex boundary with angular speed $\Gamma(n-1)/2nA$, in agreement with known results (Lamb 1975, p. 231).

Jimenez (1988) describes an alternative method for evaluating the vortex boundary which involves conformally mapping the boundary onto a circle. To the order calculated above, the two results are in agreement.

On substituting expression (2.22) for r_{vb} in (2.9) we obtain

$$\frac{\mathrm{d}\bar{z}_v}{\mathrm{d}t} = \left(\frac{\mathrm{d}W_\mathrm{E}}{\mathrm{d}z}\right)_{z_v} + \frac{\mathrm{i}A^2}{\pi\Gamma}\left[\overline{\left(\frac{\mathrm{d}^2W_\mathrm{E}}{\mathrm{d}z^2}\right)_{z_v}} + Bf_1(t)\right]\left(\frac{\mathrm{d}^3W_\mathrm{E}}{\mathrm{d}z^3}\right)_{z_v} + \frac{\mathrm{i}A^3}{12\pi^2\Gamma}\left[\overline{\left(\frac{\mathrm{d}^3W_\mathrm{E}}{\mathrm{d}z^3}\right)_{z_v}} + Df_2(t)\right]\left(\frac{\mathrm{d}^4W_\mathrm{E}}{\mathrm{d}z^4}\right)_{z_v} + O\left(\frac{\Gamma A^4}{\Lambda^7}\right), \quad (2.23)$$

where B and D are arbitrary complex constants determined by initial conditions and $f_m(t) = \exp(\mathrm{i}m\Gamma t/2A)$. Further, substituting (2.22) into (2.11) gives

$$W_\mathrm{I}(z) = \frac{\Gamma}{2\pi\mathrm{i}}\left[\ln(z-z_v) - \frac{\mathrm{i}A^2}{\pi\Gamma(z-z_v)^2}\left(\overline{\left(\frac{\mathrm{d}^2W_\mathrm{E}}{\mathrm{d}z^2}\right)_{z_v}} + Bf_1(t)\right) - \frac{\mathrm{i}A^3}{6\pi^2\Gamma(z-z_v)^3}\left(\overline{\left(\frac{\mathrm{d}^3W_\mathrm{E}}{\mathrm{d}z^3}\right)_{z_v}} + Df_2(t)\right) + O\left(\frac{A^4}{\Lambda^8}\right)\right], \quad (2.24)$$

where B, D and $f_m(t)$ are as in (2.23). The first two terms on the right-hand sides of (2.23) and (2.24) can be shown to be in agreement with the corresponding terms in the original derivation by Kida (1982); the third term represents an $O(A^3/\Lambda^6)$ correction.

When the flow field consists of N identical uniform vortices of finite core, the motion of the centroids is governed by

$$\begin{aligned}\frac{\mathrm{d}\bar{z}_v}{\mathrm{d}t} &= \frac{\Gamma}{8\pi i A^2}\oint_{R_{cv}}\left[\sum_{m\neq v}\oint_{R_{cm}}(z_{vb}-z')\ln(z_{vb}-z')\,\mathrm{d}\bar{z}'\right]\mathrm{d}\bar{z}_{vb}\\ &= \frac{\Gamma}{2\pi\mathrm{i}}\left\{\sum_{m\neq v}\frac{1}{z_{vm}} + \frac{A^2}{\pi^2}\left[\sum_{m\neq v}\frac{1}{z_{vm}^3}\left(\sum_{p\neq m}\frac{1}{\bar{z}_{mp}^2} + B_m f_1(t)\right) + \left(\sum_{m\neq v}\frac{1}{\bar{z}_{vm}^2} + B_v f_1(t)\right)\sum_{m\neq v}\frac{1}{z_{vm}^3}\right]\right.\\ &\quad\left. + \frac{A^3}{2\pi^3}\left[\left(\sum_{m\neq v}\frac{1}{\bar{z}_{vm}^3} + D_v f_2(t)\right)\sum_{m\neq v}\frac{1}{z_{vm}^4} - \sum_{m\neq v}\frac{1}{z_{vm}^4}\left(\sum_{p\neq m}\frac{1}{\bar{z}_{mp}^3} + D_m f_2(t)\right)\right] + O\left(\frac{A^4\Gamma}{\Lambda^9}\right)\right\}\end{aligned} \quad (2.25)$$

($v = 1, 2, \ldots, N$) in view of (2.23) and (2.24) with $z_{jk} = z_j - z_k$. The shape of the vth vortex is given by (2.22) where

$$r_{vb} = \left(\frac{A}{\pi}\right)^{\frac{1}{2}}\left(1 + \frac{A}{\pi}\mathrm{Im}\left[\left(\sum_{n\neq v}\frac{\mathrm{i}\,\mathrm{e}^{2\mathrm{i}\theta}}{(z_v-z_n)^2}\right) + \sum_{\substack{m\\|m|\neq 0,1}} c_m\,\mathrm{e}^{\mathrm{i}(m\theta-(m-1)\Gamma t/2A)}\right] - \frac{1}{2}\left(\frac{A}{\pi}\right)^{\frac{3}{2}}\mathrm{Im}\left[\left(\sum_{n\neq v}\frac{\mathrm{i}\,\mathrm{e}^{3\mathrm{i}\theta}}{(z_v-z_n)^3}\right) + \sum_{\substack{m\\|m|\neq 0,1}} d_m\,\mathrm{e}^{\mathrm{i}(m\theta-(m-1)\Gamma t/2A)}\right] + O\left(\frac{A^2}{\Lambda^4}\right)\right). \quad (2.26)$$

The oscillatory terms in (2.26) have a period of $O(A/\Gamma)$ which is short compared with the timescale of $O(\Lambda^2/\Gamma)$ associated with the external flow. Note that to $O(A^3/\Lambda^6)$, only the terms corresponding to $m = 2$ and 3 in (2.26) affect the motion of the centroid, causing it to oscillate rapidly about a mean position. In steady flow these terms are absent. Further, perturbation modes corresponding to these fast oscillations are stable. Henceforth, therefore, we consider motion in which these terms are absent. Such considerations hold provided the core area is small enough. For large-area vortices, when the timescale of these oscillations is comparable to that of the external flow, the oscillatory terms would need to be taken into account.

Professor D. W. Moore has pointed out that in the case of steady motion, equation (2.23) to $O(A^2/\Lambda^4)$ can also be derived by noting in (2.9) that to that order the shape of the vortex is elliptical, the ratio of the length of the axes being related to the external strain rate experienced at the centroid (Moore & Saffman 1971). Details of this derivation, due to him, are given in Appendix B. This derivation also serves to illustrate that the result (2.23) to $O(A^2/\Lambda^4)$ is consistent with the 'second-order model' of Melander *et al.* (1986) which is based on the assumption that the shape of a vortex in a strain field can be approximated by an ellipse of appropriate aspect ratio and orientation and on retaining nonlinear terms in equation (B 7) for the strain rate. In an asymptotic expansion of the type presented here, these nonlinear terms imply higher-order contributions to the velocity and for consistency such terms cannot be retained in (B 7). However, Melander *et al.* show that retaining these terms in their model gives, at least for two vortices, a good approximation to the velocity of the vortices. A quantitative comparison between the present results and those of Melander *et al.* is presented in §3.

3. A polygon of finite vortices

Here we consider the motion of N identical vortices of finite core positioned such that their centroids lie, regularly spaced, on the circumference of a circle of radius a. The vortices move along the circle with an angular velocity Ω_N. Thus with the origin at the centre of the circle, the positions of the centroids at time t are given by

$$z_v \equiv Z_v = a\,\mathrm{e}^{2\pi\mathrm{i}(v-1)/N+\mathrm{i}\Omega_N t} \quad (v = 1,2,\ldots,N). \tag{3.1}$$

This satisfies the equation of motion (2.14) provided that

$$\Omega_N = \frac{\Gamma}{2\pi a^2}(\Omega_{N0}+\alpha^2\Omega_{N2}+\alpha^3\Omega_{N3}+O(\alpha^4)), \tag{3.2}$$

where

$$\left.\begin{aligned} \Omega_{N0} &= \sum_{n=1}^{N-1}\frac{1}{1-E_n} = \frac{N-1}{2}, \\ \Omega_{N2} &= \sum_{p=1}^{N-1}\frac{1}{(1-\bar{E}_p)^2}\sum_{n=1}^{N-1}\frac{1+E_n^2}{(1-E_n)^3} = \frac{(N-1)^2(N-5)^2}{144}, \\ \Omega_{N3} &= \frac{1}{2}\sum_{p=1}^{N-1}\frac{1}{(1-\bar{E}_p)^3}\sum_{n=1}^{N-1}\frac{1-E_n^3}{(1-E_n)^4} = \frac{(N-1)^2(N-3)^2}{128}, \\ E_n &= \mathrm{e}^{2\pi\mathrm{i}n/N}, \quad \alpha = A/\pi a^2. \end{aligned}\right\} \tag{3.3}$$

The sums in (3.3) have been evaluated using the method given in Appendix C. If we put $\alpha = 0$, we recover the result for the angular velocity of a polygon of N point vortices (Havelock 1931). Further, putting $N = 1$ in (3.2) gives $\Omega_N = 0$, as required. The position of the vortex boundary is given by

$$z_{v\mathrm{b}} \equiv Z_{v\mathrm{b}} = a\,\mathrm{e}^{\mathrm{i}\psi_v}(1+(r_{v\mathrm{b}}/a)\,\mathrm{e}^{\mathrm{i}(\theta-\psi_v)}) \quad (v = 1,2,\ldots,N) \tag{3.4}$$

N	a_0	α = (core area)/πa^2 Numerical	Present	T/T_p Numerical	Present
2	0.900	0.003	0.003	1.000	1.000
	0.700	0.031	0.031	0.999	1.000
	0.500	0.105	0.105	0.998	0.999
	0.300	0.242	0.252	0.989	0.992
	0.100	0.389	0.497	0.947	0.968
3	0.900	0.003	0.003	1.000	1.000
	0.700	0.030	0.031	1.000	1.000
	0.500	0.102	0.104	0.998	0.999
	0.300	0.218	0.247	0.988	0.993
	0.100	0.244	0.494	0.943	0.974
4	0.900	0.003	0.003	1.000	1.000
	0.700	0.031	0.031	1.000	1.000
	0.500	0.104	0.106	0.999	0.999
	0.300	0.210	0.269	0.991	0.996
5	0.900	0.003	0.003	1.000	1.000
	0.700	0.031	0.031	1.000	1.000
	0.500	0.115	0.114	1.000	1.000
	0.450	0.154	0.150	0.998	0.999
6	0.900	0.003	0.003	1.000	1.000
	0.800	0.012	0.012	1.001	1.000
	0.700	0.032	0.032	1.001	1.000
	0.500	0.086	0.080	0.999	0.999

TABLE 1. Comparison between the present results and the numerical results of Dritschel (1985) for steady flow. a_0 is the ratio of the minimum to maximum distance of the boundary of a vortex from the centre. T is the period of rotation and T_p is the period of rotation of an equivalent system of point vortices of strength Γ and placed at the centroid position. For a given value of a_0, α in the 'Present' column has been obtained by iteration on equation (3.5); also, in Dritschel's notation, $\alpha = \delta^2(1+(N-1)(N-5)\delta^2/6+O(\delta^4))$ with $\delta = (1-a_0)/(1+a_0)$

where

$$r_{vb} = (A/\pi)^{\frac{1}{2}}(1-\alpha a_2\cos 2(\theta-\psi_v)+\alpha^{\frac{3}{2}}a_3\cos 3(\theta-\psi_v)+O(\alpha^2)) \tag{3.5}$$

with

$$a_2 = \frac{(N-1)(N-5)}{12}; \quad a_3 = \frac{(N-1)(N-3)}{16}; \quad \psi_v = \frac{2\pi(v-1)}{N}+\Omega_N t.$$

Note that for $N = 5$ vortices, the leading-order effect due to finite size in (3.2) and (3.5), which correspond to an elliptical deformation of the vortex, vanish; using the results of Appendix B it can be shown that this is associated with the fact that, to leading order, the strain field at a vortex due to the other vortices also vanishes.

In order that the vortices do not touch each other, we require that, approximately,

$$\alpha < \sin^2(\pi/N) \quad \text{for} \quad N > 1. \tag{3.6}$$

The vortex shapes for $\alpha/\sin^2(\pi/N) = 0.5$ for various values of N are shown in figure 1. These compare fairly well with those obtained numerically by Dritschel (1985). For a range of the ratio of minimum to maximum distance of the vortex boundary from the centre, Dritschel gives values of core area and period of rotation to three decimal places for various values of N. These are compared in table 1. The agreement is fairly good for small core area. The value of Ω_2 given by (3.2) is

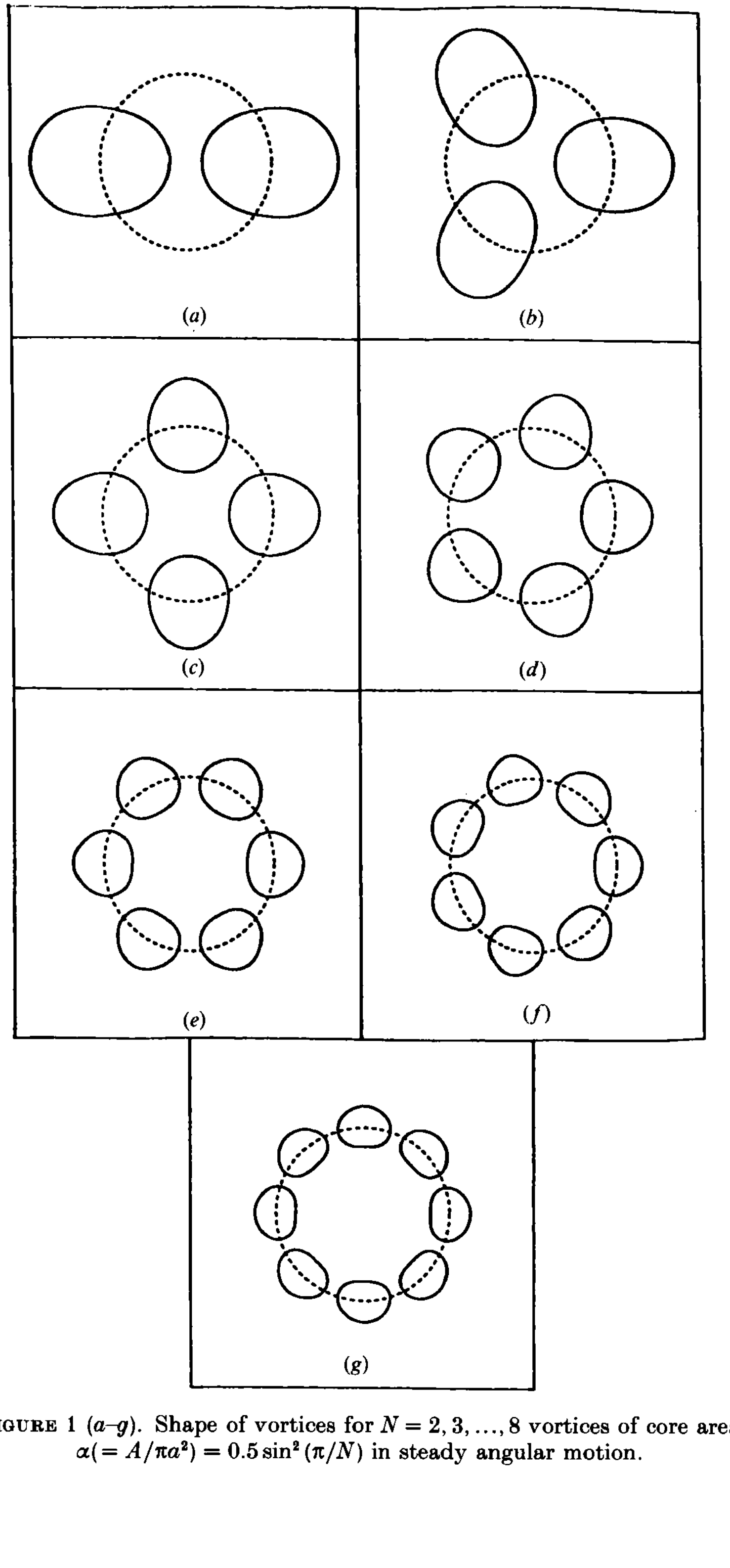

FIGURE 1 (a–g). Shape of vortices for $N = 2, 3, \ldots, 8$ vortices of core area $\alpha(= A/\pi a^2) = 0.5\sin^2(\pi/N)$ in steady angular motion.

α	Ω_{CD}/ω	Ω_{MM}/ω	Ω_2/ω	Ω_{CV}/ω
0.012272	0.003068	0.003068	0.003068	0.003068
0.060484	0.01513	0.01513	0.015128	0.015121
0.165178	0.04146	0.04146	0.0414	0.041295
0.241664	0.06108	0.06101	0.061	0.060416
0.325525	0.08344	0.08309	0.083	0.081381
0.389359	0.1027	0.1008	0.10	0.097340

TABLE 2. Comparison of angular rotation rate Ω_2 for two corotating vortices as given by (3.2) with corresponding values as given by Melander *et al.* (1986) and evaluated numerically using contour dynamics (Ω_{CD}) and their moment model (Ω_{MM}). Ω_{CV} refers to the result corresponding to a circular vortex and is given by $\Omega_{CV}/\omega = \frac{1}{4}\alpha$, which is equal to $1/\mu^2$ in the notation of Melander *et al.*

a_0	$\alpha \times 10^4$	$(\Omega_{CD}/\omega)\times 10^4$	$(\Omega_7/\omega)\times 10^4$	$(\Omega_{CV}/\omega)\times 10^4$
0.98	1.020512 44	1.5307687	1.530768 67	1.530768 67
0.96	4.168413 49	6.252624 87	6.2526206	6.2526202
0.94	9.583764 91	14.375705	14.375652	14.375647

TABLE 3. Comparison of Ω_7 as given by (3.2) with numerically evaluated values, Ω_{CD}, using contour dynamics (by courtesy of a referee) for seven corotating vortices. a_0 and Ω_{CV} are respectively as in tables 1 and 2. For a given value of a_0, α is evaluated using the formula given beneath table 1 including the $O(\delta^6)$ term, the coefficient of which for $N = 7$ vortices is 107/6

compared in table 2 with corresponding numerically evaluated values Ω_{CD} as given by Melander *et al.* (1986) and obtained using the contour integration method (that is, through numerical integration of an equivalent of equation (2.6)) and with Ω_{MM}, obtained using their second-order model. Finally, Ω_7 for fairly small vortices is compared with Ω_{CD} and table 3 (the author is grateful to a referee for supplying these latter values). The results are in fairly good agreement in both cases.

It is interesting to consider the limiting case where the radius $a \to \infty$, $N \to \infty$ with $2\pi a/N \to \Lambda$, a constant. This leads to a linear array of finite uniform vortices of spacing Λ. If we set $a = N\Lambda/2\pi$ and $v = 1$ in (3.2)–(3.4) and let $N \to \infty$, we find that

$$\Omega_N = O(1/N) \tag{3.7}$$

and

$$r_{1b} = (A/\pi)^{\frac{1}{2}}(1 + A(\pi^2/3\Lambda^2)\cos 2\phi + O(A^2/\Lambda^4)), \tag{3.8}$$

where $\phi = \theta - \frac{1}{2}\pi$. It may be noted that the $O(A^{\frac{3}{2}}/\Lambda^3)$ term in (3.8) vanishes in the limit. From (3.6) we have that $A/\Lambda^2 < 0.25\pi$ for the vortices not to touch each other. The expression for the vortex shape obtained above is in agreement with that given by Saffman & Szeto (1981) for a linear array.

4. Stability of the vortex polygon

In this section we consider the stability of the steady-state arrangement of vortices described in §3. We restrict the analysis to $O(A^2/\Lambda^4)$ here. For small-area vortices, the fast shape-deforming modes represented by the oscillatory terms in (2.26) are stable. We therefore consider these to be absent and focus attention on the displacement-type modes. For large-area vortices, the shape-deforming modes evolve on the same timescale as the motion of the centroid and must be taken into account.

Suppose that the vortices are perturbed from their steady-state positions given by (3.1) by infinitesimal two-dimensional disturbances which merely shift the vortex lines in the flow. Let the perturbed positions of the centroids of the N vortices, $z_v \equiv \tilde{z}_v$ $(v = 1, 2, \ldots, N)$ be given by

$$\tilde{z}_v = Z_v(1+\epsilon_1 z_v'), \quad \epsilon_1 \ll 1, \tag{4.1}$$

where Z_v is the unperturbed position of the vth vortex and is given by (3.1). Then on substituting this expression in (2.17) and retaining terms to $O(\epsilon_1)$, we find that $O(1)$ terms are satisfied identically in view of (3.2)–(3.5) while $O(\epsilon_1)$ terms are satisfied if

$$\frac{\partial}{\partial t}(\bar{Z}_v \bar{z}_v') = \frac{-\Gamma}{2\pi\mathrm{i}}\left[\sum_{\substack{n=1\\n\neq v}}^{N} \frac{f_{vn}}{Z_{vn}^2} + \frac{A^2}{\pi^2}\left(\sum_{\substack{n=1\\n\neq v}}^{N} \frac{1}{Z_{vn}^3} \sum_{\substack{p=1\\p\neq n}}^{N} \frac{2\bar{f}_{np}}{\bar{Z}_{np}^3}\right.\right.$$
$$\left.\left. + \sum_{\substack{n=1\\n\neq v}}^{N} \frac{3f_{vn}}{Z_{vn}^4} \sum_{\substack{p=1\\p\neq n}}^{N} \frac{1}{\bar{Z}_{np}^2} + \sum_{\substack{n=1\\n\neq v}}^{N} \frac{1}{\bar{Z}_{vn}^2} \sum_{\substack{n=1\\n\neq v}}^{N} \frac{3f_{vn}}{Z_{vn}^4} + \sum_{\substack{n=1\\n\neq v}}^{N} \frac{2\bar{f}_{vn}}{\bar{Z}_{vn}^3} \sum_{\substack{n=1\\n\neq v}}^{N} \frac{1}{Z_{vn}^3}\right)\right], \tag{4.2}$$

where

$$Z_{jk} = Z_j - Z_k = Z_j(1-E_{k-j}); \quad f_{jk} = Z_j z_j' - Z_k z_k' = Z_j(z_j' - z_k' E_{k-j})$$

and

$$E_m = \mathrm{e}^{2\pi\mathrm{i}m/N}.$$

We follow Havelock (1931) and consider the stability of the vortex polygon to the N normal modes of the disturbance. Thus we have

$$z_v' \equiv z_{vk}' = \chi\,\mathrm{e}^{\sigma t + 2\pi\mathrm{i}kv/N} \quad (k = 0, 1, 2, \ldots, N-1, \quad v = 1, 2, \ldots, N), \tag{4.3}$$

where χ is a complex constant. The stability of the system to a given mode then depends on the sign of the real part of σ.

We consider the perturbation equation for the Nth vortex. On substituting (4.3) with $v = N$ into (4.2), we find, after a little manipulation,

$$(\sigma - \mathrm{i}\Omega_N)\,\bar{\chi} = \frac{-\Gamma}{2\pi\mathrm{i}a^2}\{\chi H + \alpha^2(\chi C + 2\bar{\chi}L^2) + O(\alpha^3)\}, \tag{4.4}$$

where, with $Q_{rs} = 1 - \exp(2\pi\mathrm{i}(s+1)r/N)$ and E_m as in (4.2),

$$H = \sum_{n=1}^{N-1} \frac{Q_{nk}}{Q_{n0}^2} = \tfrac{1}{2}(k(k-N)+(N-1)),$$

$$C = 3\left(\sum_{n=1}^{N-1} \frac{1}{\bar{Q}_{n0}^2}\right)\left(\sum_{n=1}^{N-1} \frac{(1+E_n^2)\,Q_{nk}}{Q_{n0}^4}\right)$$
$$= \frac{(N-1)(5-N)}{48}(k(k-N)(k^2-Nk+4)-(N-1)(N-5)),$$

$$L = \sum_{n=1}^{N-1} \frac{\bar{Q}_{nk}}{\bar{Q}_{n0}^3} = \frac{-1}{12}(k(k-N)(2k-N-3)+(N-1)(N-5)),$$

Ω_N is given by (3.2) and α is as in (3.3). The sums in (4.4) are over the N roots of unity and were evaluated using the method given in Appendix C. The expressions were checked by evaluating the sums numerically for a number of values of $N (k = 0, 1, \ldots, N-1)$. On separating the real and imaginary parts in (4.4), it can be shown that solutions of the form (4.3) are possible provided that the following dispersion relation is satisfied:

$$\left(\frac{2\pi a^2\sigma}{\Gamma}\right)^2 = H^2 - \Omega_{N0}^2 + 2\alpha^2(HC - \Omega_{N0}D) + O(\alpha^3), \tag{4.5}$$

where $D = 2L^2 + \Omega_{N2}$, and Ω_{N0}, Ω_{N2} are given by (3.3).

Putting $\alpha = 0$ in (4.5) we recover Havelock's (1931) result for the stability of a polygon of point vortices.

The $k = 0$ mode of disturbance corresponds to a small change in the radius together with a slight rotation of the circle on which the centroids lie. As may be expected, we have from (4.5) that $\sigma^2 = 0$ in this case. Further, for $N = 1$ (so that $k = 0$), $\sigma^2 = 0$ also, as required.

The stability of the array depends on the sign of σ^2. For modes $k = 1$ and 2, the array is always stable and oscillates with frequency given by

$$\left.\begin{aligned} \left(\frac{2\pi a^2\sigma}{\Gamma}\right)^2_{k=1} &= -\frac{(N-1)^2}{4}\left(1 + \frac{\alpha^2(N-1)}{36}(8(N-2)^2 + (N-5)^2)\right), \\ \left(\frac{2\pi a^2\sigma}{\Gamma}\right)^2_{k=2} &= -(N-2)\left(1 + \frac{\alpha^2(N-1)}{144}(N(N-6)^2 + 2(N-1)^2 + 3N + 44)\right), \end{aligned}\right\} \tag{4.6}$$

respectively, for $N > 1$. In (4.5) $P_0(k,N) \equiv H^2 - \Omega_{N0}^2$ is symmetrical in k about $k = \frac{1}{2}N$ or $k = \frac{1}{2}(N+1)$ and has a maximum value, $P_{0m} = P_0(\frac{1}{2}N, N)$ or $P_0(\frac{1}{2}(N+1), N)$, according as N is even or odd. P_{0m} is negative for $N < 7$, zero for $N = 7$ and positive for $N > 7$. Thus if $\alpha = 0$, so that we have a regular polygon of N point vortices, the system is respectively stable, neutrally stable and unstable.

The term $P_2(k,N) \equiv HC - \Omega_{N0}D$ is not symmetrical in k so that σ^2 is not symmetrical in k. For $N < 7$, the maximum value of $P_2 = P_{2m} < 0$ so that finite uniform vortices of small core size on the vertices of a regular polygon are stable for these values of N.

For $N = 7$, $P_2 > 0$ for $k = 4$ and 5 modes only, it being maximum for the latter. $P_0 < 0$ for $k = 5$ so that the sign of σ^2 will depend on the value of α^2. For the maximum value of α^2 allowed (cf. (3.6)), $\sigma^2 < 0$ so that to $O(\alpha^2)$ the polygon is stable to this mode of disturbance. However for $k = 4$, $P_0 = 0$ so that $\sigma^2 > 0$ for $\alpha^2 \neq 0$. The case of σ^2 corresponding to $N = 7$ is plotted against mode number k for various values of α in figure 2(*a*) and against α^2 for different values of k in figure 2(*b*). Thus a regular polygon of seven finite-core uniform vortices is unstable.

For $N > 7$, and $k = \frac{1}{2}N$ if N is even or $k = \frac{1}{2}(N+1)$ if N is odd, $P_2 > 0$ so that since $P_{0m} > 0$ for this value of k, $\sigma^2 > 0$ for these values of N. Thus a polygon of $N \geqslant 7$ finite uniform vortices is unstable, the growth rate of disturbances being greater than in the case of point vortices.

The linear stability of a polygon of $N \leqslant 8$ finite vortices was numerically evaluated by Dritschel (1985). He considered both displacement-type disturbances as well as those which deform the vortex shape. He showed that for small-area vortices, the polygon is stable if $N < 7$ and unstable if $N = 7$ or 8 and that σ^2 for finite vortices is not symmetrical about k. These results are consistent with the above. However, for $N = 7$, unlike the present case, he obtains that the array is unstable to two displacement-type modes, corresponding to $k = 3$ and 4 in the present notation. For these modes, P_0 is zero (that is, the polygon of point vortices is neutrally stable). Thus we see from the dispersion relation (4.5) that in a numerical calculation, in order to resolve its sign and hence establish whether the system is stable or not, $(\pi a^2\sigma/\Gamma)^2$

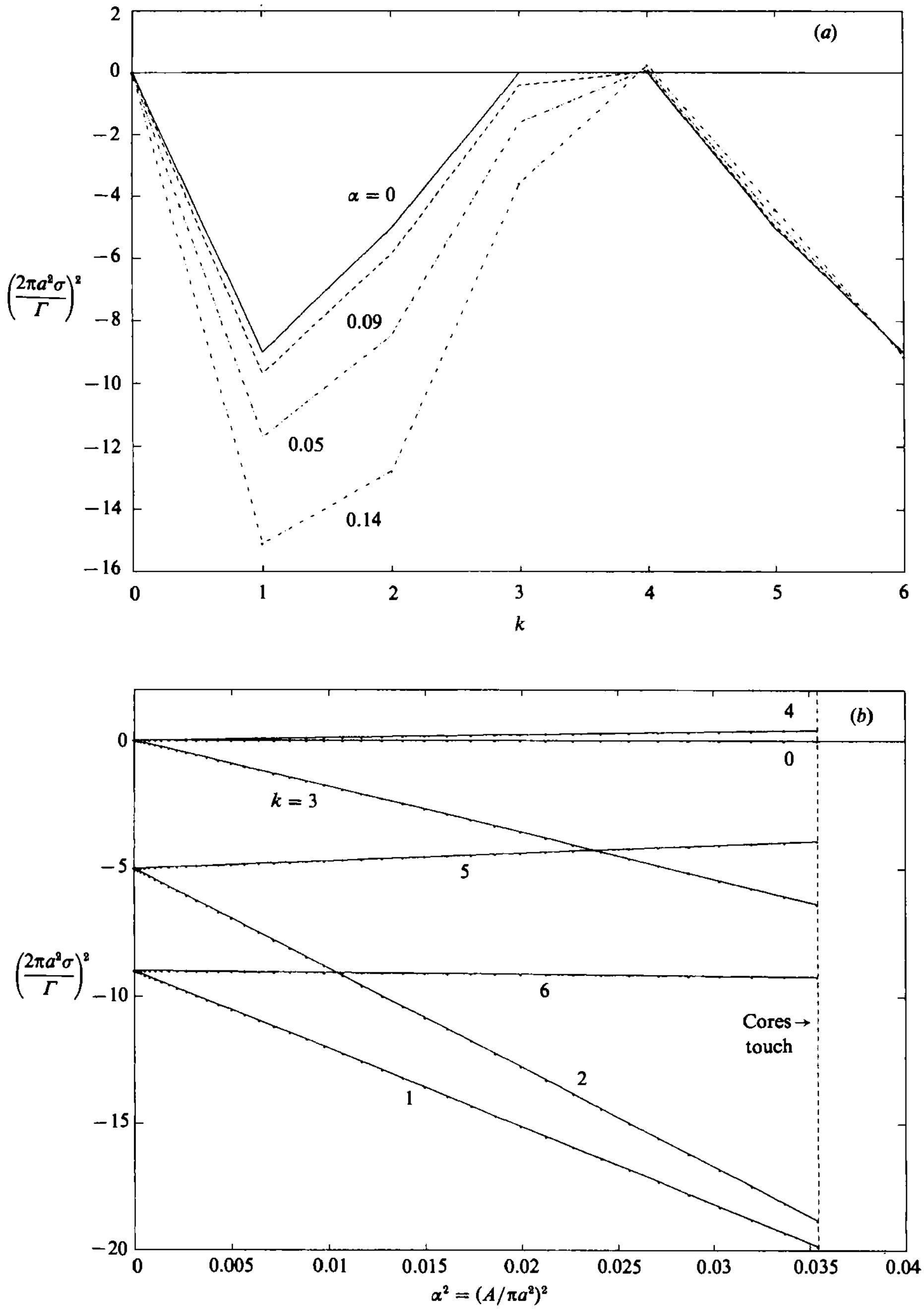

FIGURE 2. (a) Dependence of growth rate on modes of disturbance for $N = 7$ for three values of $\alpha = A/\pi a^2$. (b) Dependence of growth rate of various modes of disturbance on (core area)^2 for case $N = 7$.

needs to be evaluated to $O(\alpha^2)$. For small enough values of the core size, the magnitude of this latter term can be quite small. Dritschel does not provide information about the accuracy of his results for the specific case of $N = 7$. For $N = 8$, he states that in order to evaluate the growth rate to 4 decimal places, the steady shapes of the vortices had to be evaluated to $O(10^{-7})$ for $\alpha = O(10^{-6})$ and this appeared to place a severe strain on the computations. For the same value of α, with $N = 7$, $(\pi a^2\sigma/\Gamma)^2$ needs to be evaluated to $O(10^{-10})$ in order to resolve its sign. The corresponding requirement on the numerical accuracy of the steady shape of the vortices must therefore be enormous. If the same accuracy as in the case of $N = 8$ vortices is used, we note that this would be inadequate to resolve the sign of $(\pi a^2\sigma/\Gamma)^2$. In fact, in view of (4.5), a reliable numerical determination of stability for sufficiently small core size cannot be given for the case of $N = 7$ vortices.

For larger values of α, the requirement on the numerical accuracy of the steady shapes is apparently not so demanding and in this case comparison between the present asymptotic results and Dritschel's numerical results would suggest the existence of a threshold value of α, α_T say, such that the $k = 3$ mode is stable for $\alpha \leqslant \alpha_T$ and unstable for $\alpha \geqslant \alpha_T$. This is, however, of academic interest only since the $k = 4$ mode is, according to both results, unstable for any non-zero value of core size.

Moore (1981) has considered the representation of a circular vortex sheet by point vortices and has shown that the chaotic behaviour which sets in in a numerical calculation can be attributed to the instability of a polygon of $N > 7$ point vortices. We can examine the effect of representing the vortex sheet by N uniform vortices of small, finite core. Suppose the total strength of the sheet is $\Gamma_0 = \omega AN$ so that in (4.5) $\Gamma = \Gamma_0/N$ and $\alpha = \alpha_0/N$, where $\alpha_0 = NA/\pi a^2$. Then

$$\left(\frac{2\pi a^2\sigma}{\Gamma_0}\right)^2 = \frac{1}{N^2}\left(H^2 - \Omega_{N0}^2 + \frac{2\alpha_0^2}{N^2}(HC - \Omega_{N0}D) + O\left(\frac{\alpha_0^3}{N^3}\right)\right). \tag{4.7}$$

The most unstable mode corresponds to $k = \frac{1}{2}N$ if N is even and to $k = \frac{1}{2}(N+1)$ if N is odd. For this mode

$$\frac{2\pi a^2\sigma}{\Gamma_0} = \frac{N}{8}\left(1 + \frac{\alpha_0^2 N^2}{96}\right) + O(1) \quad \text{for} \quad N \gg 1. \tag{4.8}$$

Thus the growth rate of the most unstable mode for $N \geqslant 7$ is greater for finite uniform vortices than for point vortices and increases with N. This means that representing a circular vortex sheet by small finite-core uniform vortices, instead of point vortices, would lead to chaotic behaviour at an earlier time.

Finally, by considering the limit $a \to \infty$, $N \to \infty$ such that $2\pi a/N \to \Lambda$ as in §3, we can examine the stability of a linear array of finite uniform vortices. We put $\phi = 2\pi k/N$ and $a = N\Lambda/2\pi$ in (4.5) and let $N \to \infty$. This gives

$$\sigma^2 = \frac{\Gamma^2}{16\pi^2\Lambda^4}\phi^2(\phi - 2\pi)^2\left(1 - \frac{A^2}{3\Lambda^4}\phi(\phi - 2\pi) + O(A^3/\Lambda^6)\right). \tag{4.9}$$

The leading-order term in (4.9) gives the growth rate of disturbances to a linear array of point vortices (Lamb 1975, p. 225). Note that the pairing mode is the most unstable mode which is consistent with the prediction of Meiron *et al.* (1984). σ^2 is symmetric in ϕ about $\phi = \pi$. Saffman & Szeto (1980) presented an argument based on energy considerations which suggested that a linear array of finite uniform vortices is unstable. The above result supports this suggestion. σ^2 for this case of a linear array is plotted against ϕ in figure 3(*a*) and against A^2/Λ^4 for $\phi = \pi$ in figure 3(*b*).

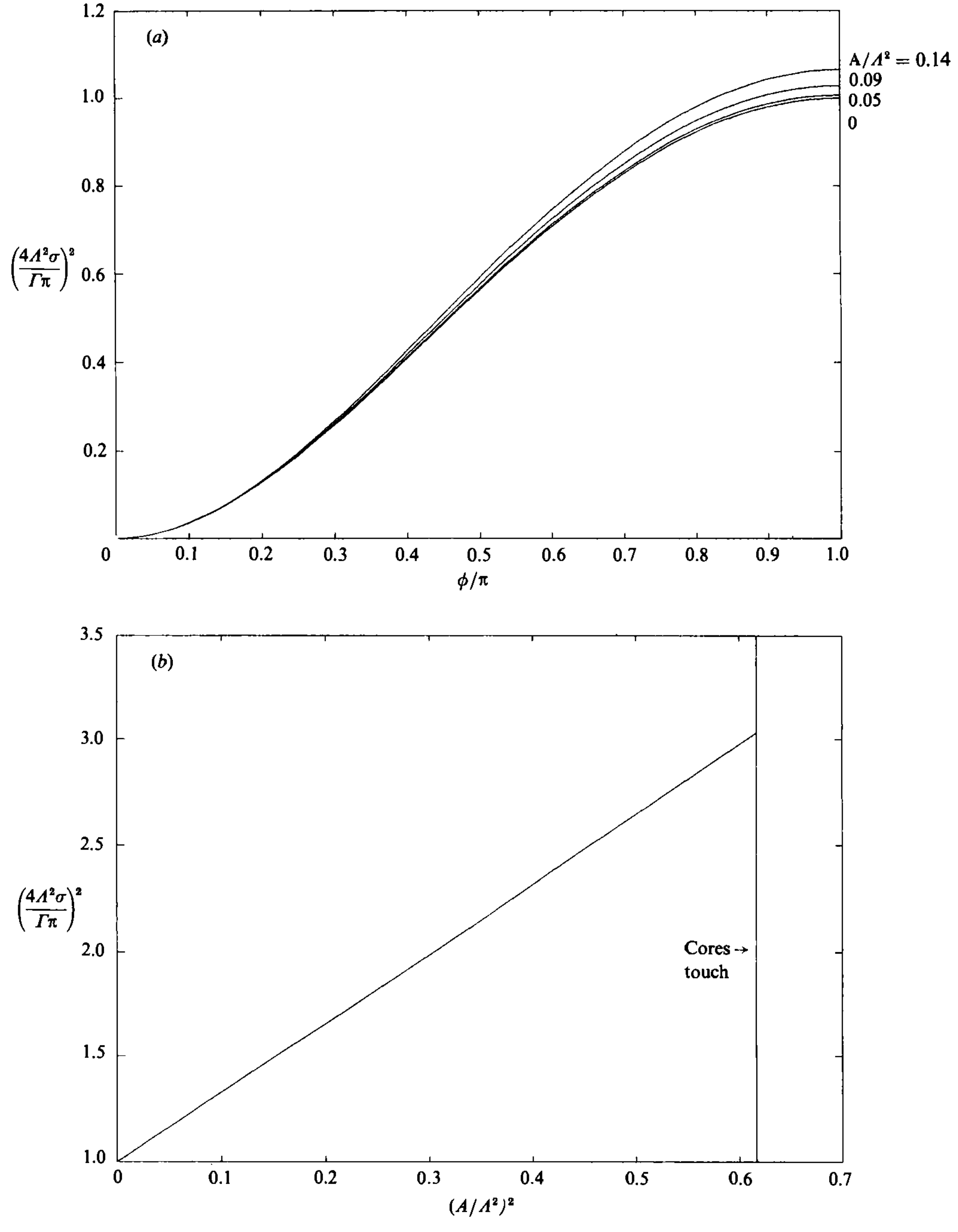

FIGURE 3. (*a*) Limiting case of a linear array. The growth rate is plotted against ϕ (see (4.9)) for various values of A/Λ^2, where Λ is the separation distance between vortices. (*b*) Limiting case of a linear array. The growth rate is plotted against $(A/\Lambda^2)^2$ for $\phi = \pi$.

5. Conclusions

Equations of motion of a uniform vortex of small but finite core subjected to an external velocity field are obtained using a complex variable formulation. The equations are used to obtain the steady configuration of N identical uniform vortices arranged in a circular row, so that the vortices are at the vertices of a regular polygon. It is shown that the circular row is stable to infinitesimal plane disturbances which displace the vortex centroids from their steady positions if $N < 7$ and unstable if $N \geqslant 7$. On comparing this result with the corresponding classical result for point vortices, we find that finite core size has a destabilizing effect on the stability of the circular row of vortices. As a limiting case, it is shown that a linear array of finite vortices in the absence of a wall is also unstable and the growth rates for small disturbances are given.

The paper is based on work carried out by the author whilst he was at Topexpress Ltd, Cambridge. The author is grateful to Professor D. W. Moore for useful discussions on certain issues raised in this work.

Appendix A. Contribution to centroid motion due to vortex-induced velocity

In this section we show that the second integral in (2.7),

$$I = \oint_{R_c} z \operatorname{Im}\left[\oint_{R_c} \ln(z-z')\,\mathrm{d}\bar{z}'\,\mathrm{d}z\right] = 0. \tag{A 1}$$

In (A 1), the branch of logarithm is chosen which makes the integrand single-valued as R_c is traversed in an anticlockwise sense. We have

$$I = \oint_{R_c} z \operatorname{Im}\left[\oint_{R_c} \frac{\partial}{\partial z}((z-z')\ln(z-z'))\,\mathrm{d}\bar{z}'\,\mathrm{d}z\right] = 0 \tag{A 2}$$

since $\oint_{R_c}\mathrm{d}\bar{z}' = 0$. Thus

$$\begin{aligned} I &= \frac{1}{2\mathrm{i}}\oint\oint_{R_c}\left(\frac{\partial}{\partial z}(z(z-z')\ln(z-z')) - (z-z')\ln(z-z')\right)\mathrm{d}\bar{z}'\,\mathrm{d}z \\ &\quad -\frac{1}{2\mathrm{i}}\oint\oint_{R_c}\left(\frac{\partial}{\partial \bar{z}}(z(\bar{z}-\bar{z}')\ln(\bar{z}-\bar{z}')) - \frac{\mathrm{d}z}{\mathrm{d}\bar{z}}(\bar{z}-\bar{z}')\ln(\bar{z}-\bar{z}')\right)\mathrm{d}\bar{z}\,\mathrm{d}z' \\ &= -\frac{1}{2\mathrm{i}}\left[\oint\oint_{R_c}(z-z')\ln(z-z')\,\mathrm{d}z\,\mathrm{d}\bar{z}' - \oint\oint_{R_c}(\bar{z}-\bar{z}')\ln(\bar{z}-\bar{z}')\,\mathrm{d}z\,\mathrm{d}z'\right] \end{aligned} \tag{A 3}$$

since the integrand is single-valued. If we denote the first and second terms on the right-hand side of (A 3) as J_1 and J_2 respectively, then

$$\begin{aligned} J_1 &= -\frac{1}{4\mathrm{i}}\oint\oint_{R_c}\left(\frac{\partial}{\partial z}((z-z')^2\ln(z-z')) - (z-z')\right)\mathrm{d}z\,\mathrm{d}\bar{z}' \\ &= 0 \end{aligned}$$

and since in J_2, z and z' are dummy variables, swapping them should not change its value. However,

$$\begin{aligned} \frac{1}{2\mathrm{i}}\oint\oint_{R_c}(\bar{z}-\bar{z}')\ln(\bar{z}-\bar{z}')\,\mathrm{d}z\,\mathrm{d}\bar{z}' &= -\frac{1}{2\mathrm{i}}\oint\oint_{R_c}(\bar{z}'-\bar{z})(\ln(\bar{z}'-\bar{z})+\ln(-1))\,\mathrm{d}z\,\mathrm{d}z' \\ &= -J_2. \end{aligned}$$

Thus $J_2 = 0$ as well.

Appendix B. Derivation of (2.23) to $O(A^2/\mathcal{A}^4)$ for steady flow using Moore & Saffman's (1971) results

Professor D. W. Moore has shown that equation (2.23) to $O(A^2/\mathcal{A}^4)$ for steady flow can be derived using the results of Moore & Saffman (1971). His calculation is presented here.

From (2.9), after an integration by parts we have that to leading order

$$\frac{\mathrm{d}\bar{z}_v}{\mathrm{d}t} = W'_{\mathrm{E}}(z_v)+\frac{\mathrm{i}}{12A}W'''_{\mathrm{E}}(z_v)M(z_v); \quad M(z_v) = \oint_{R_c}(z_{v\mathrm{b}}-z_v)^3\,\mathrm{d}\bar{z}_{v\mathrm{b}}, \tag{B 1}$$

where a prime denotes differentiation with respect to z and the path of the integral is traversed in an anticlockwise sense.

We approximate the steady shape of the vortex by an ellipse with origin at z_v and orientated so that the major axis subtends an angle ϕ to the x-axis. Thus if a and b are respectively the lengths of the semi-major and semi-minor axes of the ellipse and θ is the azimuthal angle measured anticlockwise from the major axis, the boundary of the vortex is given by

$$z_{v\mathrm{b}}-z_v = \mathrm{e}^{\mathrm{i}\phi}(a\cos\theta+\mathrm{i}b\sin\theta). \tag{B 2}$$

Substituting (B 2) into (B 1) and noting that

$$\mathrm{d}\bar{z}_{v\mathrm{b}} = -\mathrm{e}^{\mathrm{i}\phi}(a\sin\theta+\mathrm{i}b\cos\theta)\,\mathrm{d}\theta$$

and area $A = \pi ab$, it can be shown that

$$M = -\tfrac{3}{2}\mathrm{i}A(a^2-b^2)\,\mathrm{e}^{2\mathrm{i}\phi}. \tag{B 3}$$

Suppose that $a/b = 1+\eta$. Then

$$M = -\frac{3\mathrm{i}A^2}{\pi}\eta\,\mathrm{e}^{2\mathrm{i}\phi}+O(\eta^2). \tag{B 4}$$

It now remains to determine η and ϕ from the calculation of Moore & Saffman.

Suppose that the uniform vortex is subjected to a plane strain with rate e_0 so that the complex potential due to the external field is given by

$$W_{\mathrm{E}2} = \tfrac{1}{2}\mathrm{i}\,e_0\,z_1^2+Cz_1+D, \tag{B 5}$$

where

$$z-z_v = z_1\,\mathrm{e}^{\mathrm{i}\phi} \tag{B 6}$$

and C and D are constants. Moore & Saffman show that

$$\eta = 4\,e_0/\omega, \tag{B 7}$$

where ω is the uniform vorticity (although the constants C and D are zero in their calculation, a non-zero value of C represents a uniform translation and the problem can equivalently be considered in a moving frame of reference). Expanding $W_{\mathrm{E}}(z)$ in (2.2) about z_v and using (B 6), we have

$$W_{\mathrm{E}}(z) = W_{\mathrm{E}}(z_v)+z_1\,\mathrm{e}^{\mathrm{i}\phi}\,W'_{\mathrm{E}}(z_v)+\tfrac{1}{2}z_1^2\,\mathrm{e}^{2\mathrm{i}\phi}W''_{\mathrm{E}}(z_v)+\dots. \tag{B 8}$$

To $O(z_1^2)$, this must agree with $W_{\mathrm{E}2}(z)$. Thus, comparing the coefficient of z_1^2,

$$e_0\,\mathrm{e}^{2\mathrm{i}\phi} = \mathrm{i}\overline{W}''_{\mathrm{E}}(z_v). \tag{B 9}$$

Thus

$$M = \frac{12A^2}{\pi\omega}\,\overline{W}''_{\mathrm{E}}(z_v). \tag{B 10}$$

Hence, using $\Gamma = \omega A$, and substituting (B 10) into (B 1) we obtain (2.14) with terms of $O(A^3/\Lambda^6)$ omitted.

Appendix C. Sum over roots of unity

Here we present the method used in evaluating sums of the form

$$\sum_{n=1}^{N-1} \frac{E_n^q}{(1-E_n)^k} \quad (q = 0, 1, 2, \ldots, k, \ldots, N-1), \tag{C 1}$$

where

$$E_n = \exp(2\pi i n/N). \tag{C 2}$$

Consider the contour integral

$$I_N = \frac{1}{2\pi i}\oint_C \frac{-Z^q N Z^{N-1}}{(1-Z)^k (1-Z^N)} \mathrm{d}Z, \tag{C 3}$$

where the path C is chosen to enclose the unit circle and is prescribed in an anticlockwise sense. Inside C, I_N has poles at $Z = E_n$ $(n = 1, 2, \ldots, N-1)$ with residues r_n given by

$$\left.\begin{aligned} r_n &= \frac{E_n^q}{(1-E_n)^k}; \quad n \neq 0, \\ r_o &= \frac{(-1)^{k+1}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}Z^k}\left(-NZ^{q+N-1}\Big/\sum_{p=0}^{N-1} Z^p\right)_{Z=1}, \end{aligned}\right\} \tag{C 4}$$

so that

$$I_N = \sum_{n=1}^{N-1} r_n + r_o. \tag{C 5}$$

But since on C, $Z \neq 0$,

$$I_N = \frac{-1}{2\pi i}\oint_C \frac{NZ^{q-k-1}}{(1/Z-1)^k (1/Z^N-1)} \mathrm{d}Z = \frac{-1}{2\pi i}\oint_{C'} \frac{N \mathrm{d}\zeta}{\zeta^{q-k+1}(\zeta-1)^k (\zeta^N-1)} \tag{C 6}$$

on transforming to the ζ-plane with $\zeta = 1/Z$, C' being the image of C. Inside C', the transformed integrand is analytic if $q < k$ and has a pole at $\zeta = 0$ if $q \geqslant k$ with residue s_{qk}:

$$s_{qk} = \begin{cases} \dfrac{N(-1)^{k+1}(q-1)!}{(k-1)!\,(q-k)!}, & q \geqslant k \\ 0, & q < k. \end{cases} \tag{C 7}$$

Thus combining (C 5) and (C 6)

$$\sum_{n=1}^{N-1} \frac{E_n^q}{(1-E_n)^k} = r_o - \begin{cases} \dfrac{(q-1)!}{(k-1)!\,(q-k)!}, & q \geqslant k \\ 0, & q < k. \end{cases} \tag{C 8}$$

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